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Translated by L.K.

PMM U.S.S.R., Vol.49, No.6, pp.770-777, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON USING THE MORE-ACCURATE EQUATIONS OF THIN COATINGS IN THE THEORY OF AXISYMMETRIC CONTACT PROBLEMS FOR COMPOSITE FOUNDATIONS*

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More-accurate equations describing the axisymmetric deformations of elastic, thin-walled elements (coatings) are derived using the asymptotic analysis of the solution to the first fundamental problem of the theory of elasticity for a layer. The notable difference distinguishing these relations from the classical, Kirchhoff-Love and Reissner-Timoshenko equations of flexure of plates, and their modifications /1/, is, that there are no concentrated forces at the edges of the stamp when the corresponding contact problems are solved. Moreover, the formulas obtained contain the equations of classical theory as a special case. The solutions obtained using various applied theories are compared with the corresponding solution obtained using the equations of the theory of elasticity, using the example of the axisymmetric contact problem of impressing a plane circular stamp into a layer lying on a Fuss-Winkler foundation. The characteristic parameters of the problem in question are computed by numerical methods.

1. As we know /2/, the solution of the equations of the theory of elasticity can be expressed, in the case of axisymmetric problems, by a single biharmonic function $\chi(r, z)$

$$\Delta^2\chi = 0 \quad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) \tag{1.1}$$

$$2Gu = -\frac{\partial^2\chi}{\partial r\partial z}, \quad 2Gw = \left[2(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi + D \tag{1.2}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[2(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi, \quad \tau_{rz} = \frac{\partial}{\partial z}\left[(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi \tag{1.3}$$

Let us consider the first boundary value problem on the equilibrium of an elastic layer of thickness $2h$, when the application of external loads deforms it symmetrically about the z-axis. We shall seek the solution of (1.1) in the form of a Hankel integral /3/

$$\chi = \int_0^\infty \xi\Phi(\xi, z) J_0(r\xi)\,d\xi, \quad \Phi = \int_0^\infty r\chi(r, z) J_0(r\xi)\,dr \tag{1.4}$$

Substituting (1.4) into (1.1) and carrying out simple mathematical reduction, we obtain

$$\Phi(\xi, z) = (c_1 \operatorname{ch}\xi z + c_2\xi z \operatorname{sh}\xi z + d_1 \operatorname{sh}\xi z + d_2\xi z \operatorname{ch}\xi z)\,\xi^{-3} \tag{1.5}$$

where c_j and d_j $(j = 1, 2)$ are functions of ξ whose form is determined from the boundary conditions of the problem in question

$$\sigma_z(r, h) = \sigma_+(r), \quad \tau_{rz}(r, h) = \tau_+(r) \tag{1.6}$$
$$\sigma_z(r, -h) = \sigma_-(r), \quad \tau_{rz}(r, -h) = \tau_-(r)$$
$$\sigma_z, \ \tau_{rz} \to 0, \quad (r^2 + z^2) \to \infty$$

*Prikl.Matem.Mekhan., 49,6,1010-1018,1985

We assume that the functions $\sigma_\pm(r)$ and $\tau_\pm(r)$ in (1.6) satisfy the following conditions: in any finite interval $(0, R)$ $\sigma_\pm(r), \tau_\pm(r)$ are functions of bounded variation and $r^3\sigma_\pm(r), r^2\tau_\pm(r) \in L(0, \infty)$. Then we can write $\sigma_\pm(r), \tau_\pm(r)$ is the form of Hankel integrals /3/

$$\sigma_\pm(r) = \int_0^\infty \xi\Sigma_\pm(\xi) J_0(r\xi)\, d\xi, \quad \tau_\pm(r) = \int_0^\infty \xi T_\pm(\xi) J_1(r\xi)\, d\xi \tag{1.7}$$

Applying further the Hankel transform to (1.6) and taking into account (1.2)–(1.5), (1.7), we obtain

$$\begin{gathered}
u_r' = -\frac{1}{2G}\int_0^\infty [(c_1 + c_2 + d_2\xi z)\,\mathrm{sh}\,\xi z + \\
(d_1 + d_2 + c_2\xi z)\,\mathrm{ch}\,\xi z]\, J_0''(r\xi)\, \xi d\xi \\
w_r'' = -\frac{1}{2G}\int_0^\infty \{[d_1 - 2(1-2\nu)d_2 + c_2\xi z]\,\mathrm{sh}\,\xi z + \\
[c_1 - 2(1-\nu)c_2 + d_2\xi z]\,\mathrm{ch}\,\xi z\}\, J_0''(r\xi)\, \xi^2 d\xi \\
c_j(\xi) = D_-^{-1}\{[\Sigma_+(\xi) - \Sigma_-(\xi)]\, C_j(\xi) + [T_+(\xi) + T_-(\xi)]\, S_j(\xi)\} \\
C_1(\xi) = -2\nu\,\mathrm{ch}\,\xi h - \xi h\,\mathrm{sh}\,\xi h, \ C_2(\xi) = \mathrm{ch}\,\xi h \\
S_1(\xi) = (1-2\nu)\,\mathrm{sh}\,\xi h - \xi h\,\mathrm{ch}\,\xi h, \ S_2(\xi) = \mathrm{sh}\,\xi h \\
d_j(\xi) = D_+^{-1}\{[\Sigma_+(\xi) + \Sigma_-(\xi)]S_j^*(\xi) + \\
[T_+(\xi) - T_-(\xi)]\ C_j^*(\xi)\} \\
S_1^*(\xi) = -2\nu\,\mathrm{sh}\,\xi h - \xi h\,\mathrm{ch}\,\xi h, \ S_2^*(\xi) = \mathrm{sh}\,\xi h \\
C_1^*(\xi) = (1-2\nu)\ \mathrm{ch}\,\xi h - \xi h\,\mathrm{sh}\,\xi h, \ C_2^*(\xi) = \mathrm{ch}\,\xi h \\
D_\pm(\xi) = \mathrm{sh}\,2\xi h \pm 2\xi h
\end{gathered} \tag{1.8}$$

Simplifying the expansions (1.8) written in terms of Hankel transforms asymptotically, up to terms of order $O(\lambda^4)$ $(\lambda = ha^{-1} \to 0)$ and returning to the originals in accordance with the fact that $u \sim \tau h$, $w \sim \sigma h$ in contact problems (see e.g. the degenerate solutions for a thin layer /4/), we will write

$$\begin{gathered}
4Gh^2\frac{1}{r}[r\Delta_1 u_\pm(r)]' = -\frac{(1-\nu)h}{r}\Big\{[r(\tau_+ - \tau_-)]' - \\
\frac{6h^2}{5}[r\Delta_1(\tau_+ - \tau_-)]'\Big\} \mp 3(1-\nu)(\sigma_+ - \sigma_-) - \nu h^2\Delta_0(\sigma_+ + \sigma_-) \pm \\
(2-3\nu)h^2\Delta_0(\sigma_+ - \sigma_-) \mp \frac{(1-\nu)h}{r}\{3[r(\tau_+ + \tau_-)]' - \\
2h^2[r\Delta_1(\tau_+ + \tau_-)]'\} \\
4Gh^3\Delta_0^2 w_\pm(r) = 3(1-\nu)\Big[\sigma_+ - \sigma_- - \frac{4h^2}{3}\Delta_0(\sigma_+ - \sigma_-) + \\
\frac{11h^4}{15}\Delta_0^2(\sigma_+ - \sigma_-)\Big] + \frac{3(1-\nu)h}{r}[r(\tau_+ + \tau_-)]' - \\
\frac{(2-3\nu)h^3}{r}[r\Delta_1(\tau_+ + \tau_-)]' \pm \\
\frac{\nu h^3}{r}[r\Delta_1(\tau_+ - \tau_-)]' \pm (1-\nu)h^4\Delta_0^2(\sigma_+ + \sigma_-) \\
\Delta_i = d^2/dr^2 + r^{-1}\, d/dr - ir^{-2}\ (i = 0, 1)
\end{gathered} \tag{1.9}$$

Note that Eqs.(1.9) obtained cover radial, tensile and shear deformations, as well as deformations of the transverse flexure and compression of an elastic plate (coating).

If in the course of deriving (1.9) we use the fact that the parameter $\lambda = ha^{-1}$ is small in order to average the displacements over the thickness, we arrive at the following simplified equations of plate deformation:

$$\begin{gathered}
4Gh\Delta_1 u_* = -(1-\nu)(\tau_+ - \tau_-) + \frac{1-2\nu}{3}h^2\Delta_1(\tau_+ - \tau_-) - \\
\nu h(\sigma_+ + \sigma_-)' \\
\frac{4}{3}Gh^3\Delta_0^2 w_* = (1-\nu)(\sigma_+ - \sigma_-) - \frac{3-2\nu}{3}h^2\Delta_0(\sigma_+ - \sigma_-) + \\
\frac{3-2\nu}{15}h^4\Delta_0^2(\sigma_+ - \sigma_-) + \frac{(1-\nu)h}{r}\Big\{[r(\tau_+ + \tau_-)]' - \frac{h^2}{3}[r\Delta_1(\tau_+ + \tau_-)]'\Big\}
\end{gathered} \tag{1.10}$$

Unlike (1.9), the above equations account only for the radial, tensile and transverse flexural deformations. Here and earlier λ is a dimensionless parameter characterizing the relative layer thickness and a is the radius of the segment of active loading of the layer, i.e. of the segment where the surface loads $\sigma_\pm(r)$ and $\tau_\pm(r)$ comprise, for example, at least 5% of their maximum values.

The expressions (1.9), (1.10) obtained represent more-accurate differential equations of deformation of thin coatings. Neglecting on the right-hand side of the second expression of

(1.10) (or (1.9)) terms of order λ and higher, we arrive at the Kirchhof-Love equation of plate flexure /1/. If on the other hand we neglect in (1.10) the term of order $O(\lambda^4)$ only, we obtain the Reissner-type equation of plate deformation /1/. Finally, neglecting on the left- and right-hand side of the second equation of (1.9) terms of order λ^2 and higher, and on the right-hand side of the first equation terms of order λ^3, (i.e. if we neglect the flexural rigidity of the coating), we obtain

$$4Gh\,\Delta_1 u_\pm = -(1-\nu)(\tau_+ - \tau_-) - \nu h(\sigma_+ + \sigma_-)' \qquad (1.11)$$
$$\sigma_+ - \sigma_- = -hr^{-1}[r(\tau_+ + \tau_-)]'$$

Expressions (1.11) represent the equations of axisymmetric deformation of the Melan cover plate. We can also obtain from (1.9), (1.10) the Fuss-Winkler and Pasternak-Vlasov equations for the foundations.

It should be stressed that when the contact problems are solved using Eqs.(1.9), equations of the theory of elasticity or Eqs.(1.10), concentrated forces do not appear at the boundaries separating the segments. As we know /5—8/ this drawback characterizes the differential equations of flexure of elastic, thin-walled elements obtained on the basis of the Kirchhoff-Love or Reissner-Timoshenko hypotheses or their modifications.

We shall present a method of proving the fact using Eqs.(1.9). Let us take a circular plate of radius a and thickness $2h$ with a load-free face. We introduce an axial load, a shearing force and a bending moment acting in the cross-section of the plate

$$T(r) = \int_{-h}^{h} \sigma_r dz, \quad Q(r) = \int_{-h}^{h} \tau_{rz} dz, \quad M(r) = \int_{-h}^{h} \sigma_r z\, dz \qquad (1.12)$$

Using the conditions of equilibrium of the plate we can show that $T(a) = Q(a) = M(a) = 0$. Therefore Eqs.(1.9) can be transformed to

$$4Gh^2\Delta_1 u_\pm = -\nu h^2\Delta_1 P \pm 3(1-\nu)Q \mp (2-3\nu)h^2\Delta_1 Q + \qquad (1.13)$$
$$(1-\nu)h\left\{r^{-1}(rT)' - \frac{6h^2}{5}[r^{-1}(rT)'']'\right\} \mp$$
$$(1-\nu)h\{3r^{-1}(rS)' - 2h^2[r^{-1}(rS)'']'\}$$
$$4Gh^3(\Delta_0 w_\pm)' = -3(1-\nu)\left(Q - \frac{4h^3}{3}\Delta_1 Q + \frac{11h^4}{15}\Delta_1^2 Q\right) +$$
$$3h(1-\nu)r^{-1}(rS)' - (2-3\nu)h^3[r^{-1}(rS)'']' \mp$$
$$\nu h^3[r^{-1}(rT)'']' \pm (1-\nu)h^4\Delta_1^2 P$$

$$P(r) = r^{-1}\int_0^r \rho(\sigma_+ - \sigma_-)d\rho, \quad S(r) = r^{-1}\int_0^r \rho(\tau_+ + \tau_-)d\rho \qquad (1.14)$$

Solving the system of four Eqs.(1.13) we can confirm that the functions (1.12), (1.14) will be continuous provided that the functions $u_\pm(r)$ and $w_\pm(r)$ are piecewise continuous.

For example, eliminating Q and S from the equations, we obtain

$$-\alpha_1^2\Delta_1[r^{-1}(rT)'] + r^{-1}(rT)' = \psi(r)$$
$$\alpha_1^2 = \frac{6h^2}{5}\frac{(1-\nu)^2}{1-2\nu};\quad \psi = \frac{2G(1-\nu)}{1-2\nu}\left[h\Delta_1(u_+ + u_-) + \frac{\nu}{1-\nu}(w_+ - w_-)'\right]$$
$$\Delta_1 P = \frac{\nu}{(1-\nu)h}r^{-1}(rT)' + \frac{2G}{(1-\nu)h}(w_+ - w_-)'$$

which shows that T, P will be continuous provided that $u_\pm, w_\pm$ are piecewise continuous functions.

Therefore, when $r = a$, there will be no concentrated forces. It is also important to note that if the functions $u_\pm(r)$ and $w_\pm(r)$ have first-order discontinuities, then so will the stresses $\sigma_\pm(r)$ and $\tau_\pm(r)$ and vice versa, and the discontinuities will be connected with each other by the relations

$$4Gu_\pm = \frac{6h}{5}(1-\nu)(\tau_+ - \tau_-) \pm 2h(1-\nu)(\tau_+ + \tau_-)$$
$$4Gw_\pm = \frac{11h}{5}(1-\nu)(\sigma_+ - \sigma_-) \pm h(1-\nu)(\sigma_+ + \sigma_-)$$

Eqs.(1.10) have the same properties.

2. We will illustrate the use of the above equations by considering an axisymmetric problem of frictionless imbedding of a parabolic stamp, under a force P, into an elastic (G, ν) layer of thickness $2h$, lying on a Fuss-Winkler foundation, with a coating coefficient l. We shall describe the physico-mechanical properties of the layer using the equations: 1) of the theory of elasticity; 2) (1.9); 3) (1.10); 4) of Reissner-type plates

$$4Gh^3\Delta_0^2 w_* = 3(1-\nu)(\sigma_+ - \sigma_-) - (3-2\nu)h^2\Delta_0(\sigma_+ - \sigma_-) \qquad (2.1)$$

and 5) of Kirchhoff-Love plates

$$4Gh^3\Delta_0^2 w_* = 3(1-\nu)(\sigma_+ - \sigma_-) \tag{2.2}$$

The problems under consideration can be reduced, using the method of integral transforms /4/, to finding the unknown contract pressures under the stamp, from the following integral equations of the first kind, written in dimensionless form as follows:

$$\int_0^1 \sigma(\rho)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho = \lambda(\delta - \gamma r^2) \quad (0 \leqslant r \leqslant 1) \tag{2.3}$$

$$k(\zeta, z) = \int_0^\infty K(u) u J_0(u\zeta) J_0(uz)\, du \quad \left(\zeta = \frac{\rho}{\lambda},\ z = \frac{r}{\lambda}\right)$$

$$1)\quad K(u) = \frac{\operatorname{ch} 4u - 1 + nu(\operatorname{sh} 4u + 4u)}{[\operatorname{sh} 4u + 4u + nu(\operatorname{ch} 4u - 1 - 8u^2)]\, u} \tag{2.4}$$

$$2)\quad K(u) = \frac{3(1+n) + 4(1+n)u^2 + (^{16}/_5)(^{11}/_{16} + n)u^4}{3 + 4u^2 + 4(^4/_5 + n)u^4}$$

$$3)\quad K(u) = \frac{n[3 + mu^2 + (^1/_5)mu^4]}{3 + mu^2 + 4[n + (^1/_{20})m]u^4}$$

$$4)\quad K(u) = \frac{n(3 + mu^2)}{3 + mu^2 + 4nu^4}, \qquad 5)\quad K(u) = \frac{3n}{3 + 4nu^4}$$

$$\lambda = ha^{-1},\ n = G[hl(1-\nu)]^{-1},\ m = (3-2\nu)(1-\nu)^{-1}$$

which must be supplemented with the condition of statics

$$P(1-\nu)(Ga^2)^{-1} = R = 2\pi \int_0^1 \sigma(r) r\, dr \tag{2.5}$$

Further, using the approach described in /4/, we transform the Eq.(2.3), (2.4), in the case of problem 1) to the equivalent integral equation of the first kind with a difference kernel

$$\int_{-1}^{1} \varphi(\tau) k\left(\frac{\tau - t}{\lambda}\right) d\tau = \pi\lambda g(t) \quad (|t| \leqslant 1) \tag{2.6}$$

$$k(z) = \int_0^\infty L(u) \cos uz\, du, \quad K(u) = u^{-1}L(u)$$

$$g(t) = \frac{d}{dt} \int_0^t \frac{r(\delta - \gamma r^2)}{\sqrt{t^2 - r^2}}\, dr = \delta - 2\gamma t^2 \tag{2.7}$$

$$\sigma(r) = -\frac{2}{\pi} \frac{d}{dr}\left[r \int_r^1 \frac{\varphi(\tau)\, d\tau}{\tau\sqrt{\tau^2 - r^2}}\right] \tag{2.8}$$

We obtain the solution of (2.6) in accordance with the Krein formula /9/

$$\varphi(t) = \frac{1}{2M'(1)}\left[\frac{d}{db}\int_{-b}^{b} q(s,b) g(s)\, ds\right] q(t,1) - \tag{2.9}$$

$$\frac{1}{2}\int_{|t|}^{1} q(t,\xi) \frac{d}{d\xi}\left[\frac{1}{M'(\xi)} \frac{d}{d\xi} \int_{-\xi}^{\xi} q(s,\xi) g(s)\, ds\right] d\xi \quad (b = 1, |t| \leqslant 1)$$

$$\int_{-s}^{s} q(\tau, s) k\left(\frac{\tau - t}{\lambda}\right) d\tau = \pi\lambda \quad (0 \leqslant s \leqslant 1) \tag{2.10}$$

$$M(\xi) = \int_0^\xi q(t,\xi)\, dt$$

To find the unknown function $q(t, s)$ from the integral Eq.(2.10), we approximate $L(u)$ in the expression for its kernel (2.6), according to the first formula of (2.4), by the expression

$$L(u) = \frac{[(u^2 + \varepsilon^2)(u^2 + A^2B^4)]^{1/2}}{u^2 + B^2} \quad (\varepsilon \to 0,\ A = 1 + n) \tag{2.11}$$

The constant B is obtained in such a manner that the percentage divergence of the approximation (2.11) from the function $L(u) = uK(u)$ (2.4) is the smallest for all $0 \leqslant u < \infty$.

We shall seek the principal (zero) term of the asymptotic form of $q(t, s)$ for $\lambda \ll 1$ in the form /4/

$$g_\varepsilon(t,s)=\mu(\lambda)\,\omega\left(\frac{s+t}{\lambda}\right)\omega\left(\frac{s-t}{\lambda}\right)v^{-1}\left(\frac{t}{\lambda}\right) \tag{2.12}$$

where $\omega(t)$ and $v(t)$ satisfy the Eqs.(1.25), (1.26) of /10/, with the corresponding functions within the inner integrals replaced by the expression (2.11). Substituting the solutions $\omega(t)$ and $v(t)$ obtained in this manner into (2.12) and passing to the limit as $\varepsilon\to 0$, we write

$$g(t,s)=\mu(\lambda)\,\psi\left(\frac{s+t}{\lambda}\right)\psi\left(\frac{s-t}{\lambda}\right) \tag{2.13}$$

$$\psi(t)=\exp\left(-\frac{AB^2t}{2}\right)\left[I_0\left(\frac{AB^2t}{2}\right)(1+Bt)+BtI_1\left(\frac{AB^2t}{2}\right)\right]$$

Here $I_\nu(t)$ are modified Bessel functions. The corrective multiplier $\mu(\lambda)$ is obtained form the first relation of (2.10) at $t=0$ and $s=1$ after substituting (2.13) into it. We note that expression (2.11) approximates the corresponding function $L(u)$ at $n<4$ with an error not exceeding 20%. The error of the solution obtained on this basis does not exceed the error of the approximation and is reduced by introducing the corrective multiplier $\mu(\lambda)$ /4/.

The unknown radius of the region of contact between the stamp and the layer, is found from the condition $\sigma(1)=0$ which can be written, according to (2.7)–(2.9), in the form

$$\delta M_1'(1)-\gamma M_2'(1)=0 \tag{2.14}$$

$$M_1(x)=\frac{1}{2}\int_{-x}^{x}g(s,x)\,ds=\lambda\mu(\lambda)\left\{\frac{x^2}{A\lambda^2}+\right. \tag{2.15}$$

$$\left.\frac{2AB-1}{A^2B^2}\cdot\frac{x}{\lambda}+\frac{(1-AB)^2}{2A^3B^4}\left[1-\exp\left(-\frac{AB^2x}{\lambda}\right)\right]\right\}$$

$$M_2(x)=\int_{-x}^{x}g(s,x)\,s^2\,ds=\lambda^3\mu(\lambda)\left\{\frac{x^4}{2A\lambda^4}+\frac{2AB-1}{A^2B^2}\frac{x^3}{\lambda^3}+\right.$$

$$\frac{1}{AB^2}\left(\frac{1}{2}-\frac{3}{AB}+\frac{2}{A^2B^2}\right)\frac{x^2}{\lambda^2}+$$

$$\frac{1}{A^2B^4}\left(-\frac{1}{2}+\frac{4}{AB}-\frac{3}{A^2B^2}\right)\frac{x}{\lambda}+$$

$$\frac{1}{A^3B^6}\left(\frac{1}{2}-\frac{3}{AB}+\frac{9}{4A^2B^2}\right)\left[1-\exp\left(-\frac{2AB^2x}{\lambda}\right)\right]+$$

$$\exp\left(-\frac{2AB^2x}{\lambda}\right)\left[\frac{1}{A^2B^4}\left(-\frac{1}{2}+\frac{2}{AB}-\frac{3}{2A^2B^2}\right)\frac{x}{\lambda}+\right.$$

$$\left.\left.\frac{1}{AB^2}\left(-\frac{1}{2}+\frac{1}{AB}-\frac{1}{2A^2B^2}\right)\frac{x^2}{\lambda^2}\right]\right\}$$

Finally, substituting (2.8), (2.9), (2.13) into the condition of equilibrium (2.5), we obtain

$$R=4\,[\delta M_1(1)-\gamma M_2(1)]$$

Let us now consider problems 2) and 3). Since in these cases the kernel $K(u)$ (2.4) of the integral Eq.(2.3) represents a rational function, the equation can be solved in closed form. Indeed, let us write the kernel (2.3) in the form /4/

$$k(\zeta,z)=c\lambda^2\delta^*(\zeta,z)+\sum_{k=1}^{2}\gamma_k s_k\begin{cases}I_0(\zeta\gamma_k)K_0(z\gamma_k) & (\rho<r)\\ K_0(\zeta\gamma_k)I_0(z\gamma_k) & (\rho>r)\end{cases} \tag{2.16}$$

$$s_k=c\,\frac{(z_1^2-\gamma_k^2)(z_2^2-\gamma_k^2)}{(\gamma_j^2-\gamma_k^2)\gamma_k}\qquad(j=1,2;\ k\neq j)$$

$c=(4A-5\,4)(5A-1)^{-1}$, $A=1+n$ (problem 2))

$c=mA(20A+m)^{-1}$, $A=n$ (problem 3))

Here $\gamma_k=-i\delta_k$, $z_k=-i\zeta_k$, δ_k and ζ_k denote, respectively, the poles and zeros of the functions $K(u)$ (2.4) 2), 3), lying in the uper half-plane. The function $\delta^*(\zeta,z)$ in (2.16) corresponds to the singular part of the kernel and is equal to

$$\delta^*(\zeta,z)=\int_0^\infty uJ_0(u\zeta)J_0(uz)\,du$$

It has the basic property of the delta function

$$\int_0^a\delta^*(\zeta,z)\,z\varphi(z)\,dz=\varphi(\zeta)\qquad(0\leqslant\zeta\leqslant a)$$

We shall seek the solution of (2.3) in the form

$$\sigma(r)=\lambda\lim\,[\delta\sigma(\varepsilon,r)+2\gamma\partial^2\sigma(\varepsilon,r)/\partial\varepsilon^2]\quad(\varepsilon\to 0) \tag{2.17}$$

where $\sigma(\varepsilon, r)$ is a solution of (2.3) with a perturbed right-hand side, $I_0(\varepsilon r)$, which can be written in the form /4/

$$\sigma(\varepsilon, r) = X_0 J_0(\varepsilon r) + \sum_{j=1}^{2} X_j I_0\left(\frac{z_j}{\lambda}\right) \tag{2.18}$$

Substituting into the integral Eq.(2.3) with right-hand side $J_0(\varepsilon r)$ the functions $k(\zeta, z)$ and $\sigma(\varepsilon, r)$ (2.16), (2.18) and equating in the resulting relation the coefficients of $J_0(\varepsilon r)$ and $I_0(z_j r/\lambda)$ on the left and right sides, we obtain the following system of linear algebraic equations for X_j:

$$a_{1j}X_1 + a_{2j}X_2 = b_j X_0 \quad (j = 1,\ 2),\quad X_0 = \lambda^{-2}A^{-2} \tag{2.19}$$

$$a_{kj} = \frac{1}{z_k^2 - \gamma_j^2}\left[z_k K_0(g_j) I_1(p_k) + \gamma_j K_1(g_j) I_0(p_k)\right]$$

$$b_k = \frac{1}{\gamma_k^2 + \varepsilon^2\lambda^2}\left[\gamma_k K_1(g_k) J_0(\varepsilon) - \lambda\varepsilon K_0(g_k) J_1(\varepsilon)\right]$$

$$g_k = \frac{\gamma_k}{\lambda},\quad p_k = \frac{z_k}{\lambda}$$

Solving system (2.19) and using the formulas (2.17), (2.18), we can write

$$\lambda A\sigma(r) = \delta - \gamma r^2 + \frac{1}{D}\Big\langle\Big\{\frac{\delta-\gamma}{\gamma_1}K_1(g_1) - \frac{2\gamma\lambda}{\gamma_1^2}\left[K_0(g_1) + \frac{2\lambda}{\gamma_1}K_1(g_1)\right]\Big\}\left[a_{22}I_0(p_1 r) - a_{12}I_0(p_2 r)\right] + \Big\{\frac{\delta-\gamma}{\gamma_2}K_1(g_2) - \frac{2\gamma\lambda}{\gamma_2^2}\left[K_0(g_2) + \frac{2\lambda}{\gamma_2}K_1(g_2)\right]\Big\}\left[a_{11}I_0(p_2 r) - a_{21}I_0(p_1 r)\right]\Big\rangle,\quad D = a_{11}a_{22} - a_{12}a_{21} \tag{2.20}$$

We find the unknown radius of the area of contact between the stamp and plate from the condition $w_+^{(4)}(1) = 0$, which is equivalent to $\sigma(1) = 0$, and find the force acting on the stamp from (2.5), by substituting into it the values of the stresses (2.20).

We shall use Reissner-type equations for the plates (2.1) to describe the physico-mechanical properties of a thin layer. Then formulas (2.3), (2.4); 4) give

$$k(\zeta, z) = \sum_{k=1}^{2} \gamma_k s_k \begin{cases} I_0(\zeta\gamma_k) K_0(z\gamma_k) & (\rho < r) \\ K_0(\zeta\gamma_k) I_0(z\gamma_k) & (\rho > r) \end{cases} \tag{2.21}$$

$$s_k = \frac{m}{4}\,\frac{z_1^2 - \gamma_k^2}{(\gamma_j^2 - \gamma_k^2)\gamma_k}\quad (k, j = 1, 2;\ k \neq j) \tag{2.22}$$

Using the relations (2.21), (2.22) we find, as before, from the integral Eq.(2.3) and the condition of statics (2.5).

$$\lambda n\sigma(r) = \delta - \gamma r^2 + \frac{1}{a_{12}} I_0(p_1, r)\Big\{\frac{\delta-\gamma}{\gamma_2}K_1(g_2) - \frac{2\gamma\lambda}{\gamma_2^2}\left[K_0(g_2) + \frac{2\lambda}{\gamma_2}K_1(g_2)\right]\Big\} \tag{2.23}$$

$$\frac{\lambda n}{2\pi}R = \frac{\delta}{2} - \frac{\gamma}{4} + \frac{1}{a_{12}p_1} I_1(p_1)\Big\{\frac{\delta-\gamma}{\gamma_2}K_1(g_2) - \frac{2\gamma\lambda}{\gamma_2^2}\left[K_0(g_2) + \frac{2\lambda}{\gamma_2}K_1(g_2)\right]\Big\}$$

To find, in this case, the unknown region of contact, we must use the relation $w_*''(1) = 0$ which is equivalent to the fact that there are no concentrated forces when $r = 1$ ($Q(1) = 0$ (see (1.12)). From (2.3), (2.23) we obtain

$$\sum_{j=1}^{2}(-1)^j \frac{1}{a_{1j}}\Big\{\frac{\delta-\gamma}{\gamma_j}K_1(g_j) - \frac{2\gamma\lambda}{\gamma_j^2}\left[K_0(g_j) + \frac{2\lambda}{\gamma_j}K_1(g_j)\right]\Big\} = 0$$

Finally we consider the solution of the problem in question for the case when the mechanical properties of the layer are modelled by the Kirchhoff-Love equations of plate flexure (2.2). As was shown in /6/, the contact stresses will, in this case, have the following structure:

$$\sigma(r) = \sigma_*(r) + R_*\delta(r-1) \tag{2.24}$$

i.e. they are composed of the distributed load and concentrated forces acting along the edge of the line of contact.

Using the formulas (2.3), (2.4) 5) we obtain, for the kernel $k(\zeta, z)$ of the integral Eq.(2.3), the representation (2.21) where $s_k = (-1)^{k+1} i\sqrt{3n}\,(4\gamma_k)^{-1}$. Substituting this into (2.3) and using

(2.24) we obtain, after some reduction, the expressions

$$\lambda n\sigma_*(r)=\delta-\gamma r^2,\quad nR_*=(\delta-\gamma)\gamma_1^{-1}K_1(g_1)K_0^{-1}(g_1)-$$
$$2\gamma\lambda\gamma_1^{-2}[1+2\lambda\gamma_1^{-1}K_1(g_1)K_0^{-1}(g_1)],$$
$$R=2\pi\left[R_*+\frac{1}{\lambda n}\left(\frac{\delta}{2}-\frac{\gamma}{4}\right)\right] \tag{2.25}$$

In this case the unknown region of contact must be determined from the condition $w_*''(1)=-2\gamma$ /6, 7/ equivalent to the fact that there are no bending moments on the edge of the region of constact $(M(1)=0$ (see (1.12))

$$(\delta-\gamma)\left[\frac{1}{\gamma_1}K_1(g_1)K_0^{-1}(g_1)-\frac{1}{\gamma_2}K_1(g_2)K_0^{-1}(g_2)\right]-$$
$$2\gamma\lambda\left[\frac{1}{\gamma_1^2}-\frac{1}{\gamma_2^2}+\frac{2\lambda}{\gamma_1^3}K_1(g_1)K_0^{-1}(g_1)-\frac{2\lambda}{\gamma_2^3}K_1(g_2)K_0^{-1}(g_2)\right]=0$$

3. Let us give examples of the calculation of the mechanical characteristics of the problem in question for $\nu=0.3$; $\gamma\lambda=1.5\cdot10^{-2}$; $n=0.5^p$ $(p=\text{const})$.

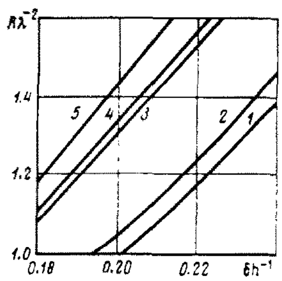

Fig.1

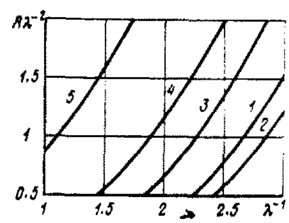

Fig.2

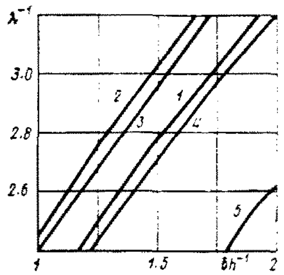

Fig.3

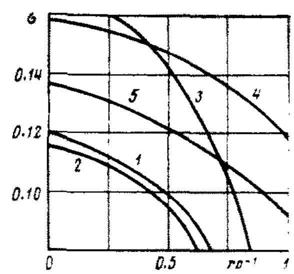

Fig.4

Fig.1 shows the dependence of the penetration of the stamp on the force applied for $p=-3$ (here and henceforth the number on the curve corresponds to the number of the problem). The dependence of the contact region on the applied force is shown in Fig.2 $(p=-1)$ from which we see that in the case of the applied theories, which disregard the effect of transverse compression, the values of λ^{-1} are too low. Fig.3 shows the relation between the penetration of the stamp and the area of the region of contact for $p=2$. The distribution of contact pressures (of distributed load in the case of problem 5) is shown for $p=0$, $R\lambda^{-2}=2.4$ in Fig.4. We note that theoreies 4 and 5 yield non-zero values for the contact pressures at the edge of the contact area, and in problem 5 concentrated forces will also appear.

Thus the results obtained show that the best approximation to problem 1 over the set of characteristics examined here over the prescribed range of values of the parameter p is given by the solution of problem 2.

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Translated by L.K.

PMM U.S.S.R., Vol.49, No.6, pp. 777-782, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON A METHOD OF SOLVING TWO-DIMENSIONAL INTEGRAL EQUATIONS OF AXISYMMETRIC CONTACT PROBLEMS FOR BODIES WITH COMPLEX RHEOLOGY*

A.V. MANZHIROV

A two-dimensional integral equatin appearing in axisymmetric contact problems for bodies with complex rheology is studied. A method of constructing the solution of this equation in proposed, based on inspecting the non-classical spectral properties of an integral operator. A contact problem for a non-uniformly aging viscoelastic foundation is solved as an example.

1. Consider the integral equation

$$c(t)(\mathbf{I}-\mathbf{L}_1)\,q(r,t)+(\mathbf{I}-\mathbf{L}_2)\,\mathbf{F}q(r,t)=\delta(t)-g(r) \tag{1.1}$$

$$(\mathbf{I}-\mathbf{L}_k)f(t)=f(t)-\int_1^t f(\tau)K_k(t,\tau)\,d\tau \quad (k=1,2) \qquad \mathbf{F}v(r)=\int_\varepsilon^1 v(\rho)\,k(\rho,r)\,\rho\,d\rho, \quad 0\leqslant\varepsilon<1$$

$$c(t)>0,\ \delta(t)\in C[1,T];\ g(r)\in L_2(\Omega);\quad q(r,t)\in L_2(\Omega)\times C[1,T]$$

with the auxilliary condition

$$P(t)=\int_\varepsilon^1 q(r,t)\,r\,dr,\quad P(t)\in C[1,T] \tag{1.2}$$

Here $K_k(t,\tau)$ are Volterra kernels /1/, the operator $\mathbf{F}$ is completely continuous, selfconjugate and positive definite and acts from $L_2(\Omega)$ into $L_2(\Omega)$, Ω is a region bounded by circles of radii ε and 1 (when $\varepsilon=0$ Ω is a unit circle), and

$$\int_\varepsilon^1\int_\varepsilon^1 k^2(\rho,r)\,r\rho\,dr\,d\rho<\infty \tag{1.3}$$

Note that the kernel of the integral operator $\mathbf{F}$ admits of the representation /2/

$$k(r,\rho)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}P_m^*(r)\,P_n^*(\rho) \tag{1.4}$$

where $P_m^*(r)$ is a complete system of functions orthonormalized in $L_2(\Omega)$. We choose this function as follows ($P_m(x)$ is the Legendre polynomial):

$$P_m^*(r)=\sqrt{\frac{4m+2}{1-\varepsilon^2}}\,P_m\left(\frac{\varepsilon^2+1-2r^2}{1-\varepsilon^2}\right)\quad (m=0,1,2,\ldots),\qquad \int_\varepsilon^1 P_m^*(r)\,r\,dr=\begin{cases}[(1-\varepsilon^2)/2]^{1/2}, & m=0\\ 0, & m\neq 0\end{cases} \tag{1.5}$$

Moreover, by virtue of (1.3) and Parseval's equation, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}^2<\infty \tag{1.6}$$

2. Let us determine in (1.1), (1.2) $q(r,t)$ and $\delta(t)$, assuming that the remaining functions are given.

We introduce a space of functions belonging to $L_2(\Omega)$ and such that their integral over Ω is zero, and denote it by $L_2^c(\Omega)$.

*Prikl.Matem.Mekhan., 49, 6, 1019-1025, 1985